FORCED CONVECTION OF A COMPRESSIBLE VISCOUS GAS NEAR A HEAT SOURCE

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PMM Vol.26, No.1, 1962, pp. 187-189

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(Received June 26, 1961)

We consider the problem of forced convection near a heat source in a homogeneous flow of viscous heat-conducting gas. The study is carried out by an approximate boundary layer theory; this means that the heat transfer by conduction in the direction of motion is neglected in comparison with convective heat transfer. It will be shown below that the problem can be reduced to the corresponding problem for an incompressible fluid by means of Dorodnitayn's transformation, provided that the viscosity temperature relation is of the form given by Chapman and Rubesin and that the Prandtl number is constant; a comparison of the isothermals concludes the note.

It is well-known that, if a heat source of intensity Q is placed in a homogeneous flow of incompressible fluid, within the accuracy of the boundary layer approximation the temperature distribution will satisfy the equation

$$\frac{\partial \vartheta}{\partial x} = \frac{a}{U} \frac{\partial^2 \vartheta}{\partial y^2} \tag{1}$$

with the boundary conditions

$$\frac{\partial \vartheta}{\partial y} = 0 \quad \text{when } y = 0, \qquad \vartheta = 0 \quad \text{when } y = \infty \tag{2}$$

and the additional condition of constant heat source intensity

$$\int_{-\infty}^{\infty} \vartheta \, dy = \frac{Q}{\rho c U T_\infty} \qquad \left(\vartheta = \frac{T - T_\infty}{T_\infty}, \quad a = \frac{k}{c\rho} = \text{const}\right) \tag{3}$$

The solution of the system (1), (2), (3) is given by the formula

$$\vartheta = \frac{1}{2\sqrt{\pi}} \frac{Q}{c\rho U T_\infty} \sqrt{\frac{U}{ax}} \exp\left(-\frac{Uy^2}{4ax}\right) \tag{4}$$

The family of isothermals T = const is determined by the one curve

$$Y = \sqrt{-X \ln X} \tag{5}$$

where

$$Y = \sqrt{2\pi} \frac{c\rho U (T - T_\infty)}{Q} y, \qquad X = \pi \frac{4a}{U} \left[\frac{c\rho U (T - T_\infty)}{Q}\right]^2 x \tag{6}$$

With the same approximations of boundary layer theory the forced convection near a heat source in a compressible flow of viscous gas is determined by the system

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \qquad \rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y} \mu \frac{\partial u}{\partial y} \tag{7}$$

$$c_p \rho \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y} k \frac{\partial T}{\partial y} + \mu\left(\frac{\partial u}{\partial y}\right)^2, \qquad \rho T = \rho_\infty T_\infty$$

with the boundary conditions

$$\frac{\partial T}{\partial y} = 0, \quad v = 0 \ \text{ when } y = 0; \qquad T = T_\infty, \quad u = U_\infty \ \text{ when } y = \infty \tag{8}$$

and the condition of constancy of heat source intensity

$$\int_{-\infty}^{\infty} c_p \rho u (T - T_\infty)\, dy = Q \tag{9}$$

Introducing the stream function by the formulas

$$\rho u = \rho_\infty \frac{\partial \psi}{\partial y}, \qquad \rho v = -\rho_\infty \frac{\partial \psi}{\partial y}$$

we turn to Dorodnitsyn's variables

$$y = \int_0^\eta \frac{\rho_\infty}{\rho} d\eta, \qquad x = \xi$$

We assume the Prandtl number to be constant and we represent the connection between viscosity and temperature by means of the function suggested by Chapman and Rubesin. Then

$$\mu\rho = C\mu_\infty \rho_\infty, \qquad k\rho = C k_\infty \rho_\infty \qquad (C = \text{const})$$

Using the second equation of the system (7), we obtain for the stream function ψ the equation

$$\frac{\partial\psi}{\partial\eta}\frac{\partial^2\psi}{\partial\xi\,\partial\eta}-\frac{\partial\psi}{\partial\xi}\frac{\partial^2\psi}{\partial\eta^2}=\frac{C\mu_\infty}{\rho_\infty}\frac{\partial^3\psi}{\partial\eta^3} \qquad (10)$$

with the boundary conditions

$$\frac{\partial^2\psi}{\partial\eta^2}=0,\quad \frac{\partial\psi}{\partial\xi}=0 \text{ when } \eta=0; \qquad \frac{\partial\psi}{\partial\eta}=U_\infty \text{ when } \eta=\infty \qquad (11)$$

The solution of the system (10), (11) is obviously $\psi = U_\infty\eta$. Consequently

$$u=U_\infty, \qquad v=-\frac{\rho_\infty}{\rho}U_\infty\frac{\partial\eta}{\partial x} \qquad (12)$$

Using the third equation of the system (7), we obtain for the temperature distribution the equation

$$\frac{\partial T}{\partial\xi}=\frac{a_\infty}{U_\infty}\frac{\partial^2 T}{\partial\eta^2} \qquad (13)$$

with the boundary conditions

$$\frac{\partial T}{\partial\eta}=0 \quad \text{при } \eta=0, \qquad T=T_\infty \quad \text{при } \eta=\infty \qquad (14)$$

and the condition of constancy of heat source intensity

$$\int_{-\infty}^{\infty} c_p\rho_\infty U_\infty T\,d\eta=Q \qquad \left(a_\infty=C\,\frac{k_\infty}{c_p\rho_\infty}\right) \qquad (15)$$

Comparison of (13) to (15) with (1) to (3) establishes the complete identity of the two problems. The solution is given by Formula (4), which in the latter case assumes the form

$$\frac{T-T_\infty}{T_\infty}=\frac{1}{2\sqrt{\pi}}\,\frac{Q}{c_p\rho_\infty U_\infty T_\infty}\sqrt{\frac{U_\infty}{a_\infty\xi}}\exp\left(-\frac{U_\infty\eta^2}{4a_\infty\xi}\right) \qquad (16)$$

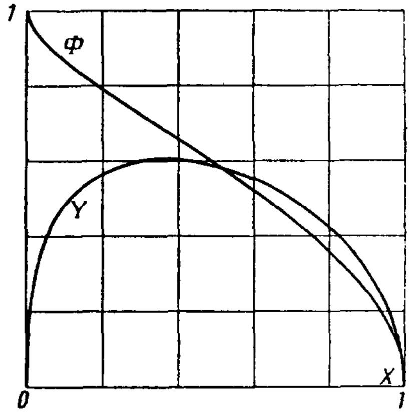
On the basis of (16) we find the connection between the variables of Dorodnitsyn η, ξ and the physical variables x, y to be given by the formulas

$$y=\eta+\frac{Q}{2c_p\rho_\infty U_\infty T_\infty}\,\Phi\left(\frac{\eta}{\sqrt{4a_\infty\xi/U_\infty}}\right), \qquad x=\xi \qquad \left(\Phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-z^2}dz\right) \qquad (17)$$

The Formulas (17) show that $y > \eta$, which means that the compressibility produces, as could be expected, an intensification of the transversal diffusion of heat. It is interesting to note that along the isothermals we have, in an incompressible fluid, $y \to \pm 0$ when $x \to 0$, i.e.

all isothermals intersect each other at the origin of the coordinates, while in the case of a compressible fluid we find along the isothermals

$$y \to \pm y_0 = \frac{Q}{2c_p\rho_\infty U_\infty T_\infty} \quad \text{when} \quad x \to 0$$



i.e. all isothermals intersect each other in points with the coordinates $(0, \pm y_0)$. We note that $2y_0$ represents the width of a homogeneous stream of gas, of density ρ_∞ and specific heat c_p, moving with the velocity U_∞ at the temperature T_∞ and losing heat at the rate Q equal to the strength of the heat source. The indicated peculiarities appearing at $x = 0$ are caused by the accepted approximation characteristics of the theory of the boundary layer. Using the Formulas (5) and (17), we obtain for the isothermals in the case of compressibility,

$$Y = \sqrt{X \ln X} + \sqrt{\frac{\pi}{2}}\,\frac{T - T_\infty}{T_\infty}\,\Phi\left(\sqrt{\frac{1}{2}\ln\frac{1}{X}}\right) \tag{18}$$

where

$$X = \pi\left[\frac{c_p\rho_\infty U_\infty (T - T_\infty)}{Q}\right]^2 \frac{4a_\infty x}{U_\infty}, \qquad Y = \sqrt{2\pi}\,\frac{c_p\rho_\infty U_\infty (T - T_\infty)}{Q}\,y$$

The figure gives a graph of the function $Y = \sqrt{(X \ln X)}$, which is the isothermal for the case of a heat source placed into a homogeneous flow of incompressible fluid, and a graph of the function $\Phi[\sqrt{(1/2 \ln X)}]$, which characterizes the correction for the case of compressibility.

An analogous study can be carried out for the case of axisymmetrical flow as well.

Translated by I.M.